# The volumes of a compact hyperbolic antiprism

Vuong Huu Bao
joint work with Nikolay Abrosimov

Novosibirsk State University

$G_2R_2$

August 6-18, Novosibirsk, 2018

# Introduction

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the main geometrical invariants for a 3-dimensional manifold.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to obtain a 3-manifold. Thus the volume of 3-manifold is the volume of prescribed fundamental polyhedron.

**Theorem (Thurston, Jørgensen)**

*The volumes of hyperbolic 3-dimensional hyperbolic manifolds form a closed non-discrete set on the real line. This set is well ordered. There are only finitely many manifolds with a given volume.*

# Introduction

1835, Lobachevsky and 1982, Milnor computed the volume of an ideal hyperbolic tetrahedron in terms of Lobachevsky function.

1993, Vinberg computed the volume of hyperbolic tetrahedron with at least one vertex at infinity.

1907, Gaetano Sforza; 1999, Yano, Cho ; 2005 Murakami; 2005 Derevnin, Mednykh gave different formulae for general hyperbolic tetrahedron.

2009, N. Abrosimov, M. Godoy and A. Mednykh found the volumes of spherical octahedron with $mmm$ or $2|m$-symmetry.

2013, N. Abrosimov and G. Baigonakova, found the volume of hyperbolic octahedron with $mmm$-symmetry.

2013, V. Krasnov found the volume of hyperbolic octahedron with $2|m$-symmetry.

2015, N. Abrosimov, E. Kudina and A. Mednykh found the volume of hyperbolic octahedron with $\overline{3}$-symmetry.

# Some motivation to find exact volume formulas

It is difficult problem to find the exact volume formulas for hyperbolic polyhedra of prescribed combinatorial type. It was done for hyperbolic tetrahedron of general type, but for general other hyperbolic polyhedron it is an open problem.

Nevertheless, if we know that a polyhedron has a symmetry, then the volume calculation is essentially simplified. Firstly this effect was shown by Lobachevskij. He found the volume of an ideal tetrahedron, which is symmetric by definition.

# Preliminaries

## Proposition

*Schläfli formula*
*Let $\mathbb{H}^3$ be a hyperbolic 3-dimensional space of constant curvature -1. Consider a family of convex polyhedra $P$ in $\mathbb{H}^3$ depending on one or more parameters in a differential manner and keeping the same combinatorial type. Then the differential of the volume $V = V(P)$ satisfies the equation*

$$dV = -\frac{1}{2} \sum_{\theta} \ell_\theta \, d\theta$$

*where the sum is taken over all edges of $P$, $\ell_\theta$ denotes the edge length and $\theta$ is the interior dihedral angle along it.*

# The *Cayley–Klein model* of the hyperbolic space $\mathbb{H}^3$

The *Cayley–Klein model* of the hyperbolic space $\mathbb{H}^3$ is the set of vectors that form the unit ball $K = \{(x_1, x_2, x_3, 1) \ : \ x_1^2 + x_2^2 + x_3^2 < 1\}$ lying in the hyperplane $x_4 = 1$. In this model hyperbolic lines and planes are represented by Euclidean lines and planes respectively. Convexity is also preserved. The *distance* $\rho(V, W)$ between vectors $V$ and $W$ is defined by the equality

$$\cosh \rho(V, W) = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle \langle W, W \rangle}}. \tag{1}$$

Consider an inner dihedral angle $\theta$ formed by two planes $P, Q$. We denote by $N, M$ the normal vectors to the planes $P, Q$ directed outwards of the dihedral angle $\theta$. Then

$$\cos \theta = -\frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \langle M, M \rangle}}. \tag{2}$$

Where $\langle \cdot, \cdot \rangle$ - inner product of the Minkowski space $\mathbb{R}^4_1$

# Antiprism $\mathcal{A}_n$

## Definition

An *antiprism* $\mathcal{A}_n$ is a convex polyhedron with two equal regular $n$-gons as the top and the bottom and $2n$ equal triangles as the lateral faces. The antiprism can be regarded as a drum with triangular sides (see Fig. where for $n = 5$ the lateral boundary is shown).

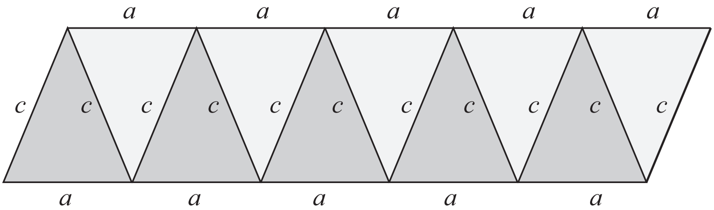



Fig.: The lateral faces of antiprism $\mathcal{A}_5$

# Antiprism $\mathcal{A}_n$

An antiprism $\mathcal{A}_n$ with $2n$ vertices has a symmetry group $S_{2n}$ generated by a mirror-rotational symmetry of order $2n$ denoted by $C_{2n\,h}$ (in Shönflies notation). In Hermann–Mauguin notation this type of symmetry is denoted by $\overline{2n}$. The element $C_{2n\,h}$ is a composition of a rotation by the angle of $\pi/n$ about an axis passing through the centres of the top and the bottom faces and reflection with respect to a plane perpendicular to this axis and passing through the middles of the lateral edges

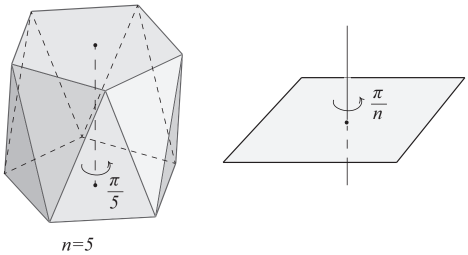



Fig.: The symmetry of an antiprism

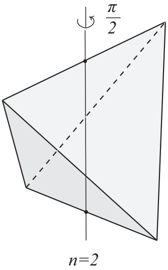
# Antiprism: ideal or compact

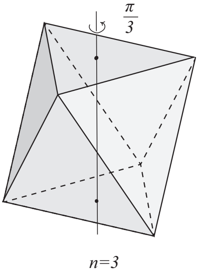
The above definitions of an antiprism $\mathcal{A}_n$ and its symmetry group $S_{2n}$ take place either for Euclidean or the hyperbolic space. By definition, $\mathcal{A}_n$ has two types of edges. Denote by $a$ the length of those edges that form top and bottom $n$-gonal faces. Set $c$ for the length of the lateral edges. Denote the dihedral angles by $A, C$ respectively. Thus, we will designate as $\mathcal{A}_n(a, c)$ for the antiprism $\mathcal{A}_n$ given by its edge lengths $a, c$.

The ideal antiprism in $\mathbb{H}^3$ with all vertices at infinity was studied by A. Yu. Vesnin and A. D. Mednykh (1995–1996). A particular case of ideal rectangular antiprism is due to W. P. Thurston (1980). In the case of ideal antiprism the dihedral angles are related by a condition $2A + 2C = 2\pi$ while in a compact case the inequality $2A + 2C > 2\pi$ holds.

# Antiprism $\mathcal{A}_n$: cases n=2 and n=3

For $n = 2$ the $n$-gons at the top and the bottom of antiprism $\mathcal{A}_n$ degenerate to corresponding two skew edges. Thus we obtain a tetrahedron with symmetry group $S_4$ (see Fig.). The volume a compact hyperbolic tetrahedron of this type was given by N. Abrosimov and B. Vuong (2017).




For $n = 3$ the antiprism $\mathcal{A}_n$ is an octahedron with symmetry group $S_6$ (see Fig.). The volume of a compact hyperbolic octahedron with this type of symmetry was found by N. Abrosimov, E. Kudina and A. Mednykh (2018).

# Compact Hyperbolic Antiprism

**Theorem 1 (Abrosimov, Vuong)**

*A compact hyperbolic antiprism $\mathcal{A}_n(a, c)$ with the symmetry group $S_{2n}$ is exist if and only if* $1 + \cosh a - 2\cosh c + 2\,(1 - \cosh c)\cos\frac{\pi}{n} < 0$.

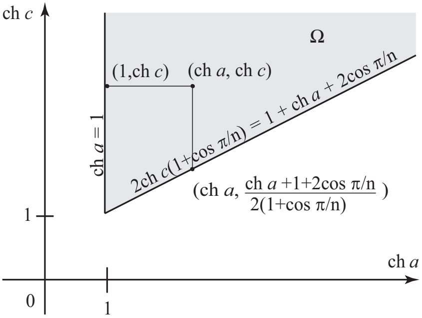



Fig.: Existence domain of a compact hyperbolic antiprism $\mathcal{A}_n$

**Theorem 2 (Abrosimov, Vuong)**

*The dihedral angles $A, C$ and the edge lengths $a, c$ of a compact hyperbolic antiprism with $2n$ vertices are related by the equalities*

$$\cos A = \frac{-\sqrt{\cosh a - 1}\,\left(1 + \cosh a - 2\cosh c\,\cos\frac{\pi}{n}\right)}{\sqrt{2(1 + \cosh a - 2\cosh^2 c)(\cos\frac{2\pi}{n} - \cosh a)}},$$

$$\cos C = \frac{\cosh c - \cosh a\,\cosh c + 2(\cosh^2 c - 1)\cos\frac{\pi}{n}}{1 + \cosh a - 2\cosh^2 c}.$$

**Theorem 3 (Abrosimov, Vuong)**

*The volume of a compact hyperbolic antiprism with* $2n$ *vertices and edge lengths* $a, c$ *is given by the formula*

$$V = n \int_{c_0}^{c} \frac{a\,G + t\,H}{(2\cosh^2 t - 1 - \cosh a)\sqrt{R}}\,dt, \text{ where}$$

$$G = 2\left(\cosh t - \cos\frac{\pi}{n}\right)\sinh a\,\sinh t,$$

$$H = -(\cosh a - 1)\left(1 + \cosh a + 2\cosh^2 t - 4\cosh t\cos\frac{\pi}{n}\right),$$

$$R = 2 - \cosh a(2 + \cosh a) + \cosh 2t + 4(\cosh a - 1)\cosh t\cos\frac{\pi}{n} - 2\sinh^2 t\cos\frac{2\pi}{n} \text{ and } c_0 \text{ is the root of the equation}$$

$$2\cosh c\left(1 + \cos\frac{\pi}{n}\right) = 1 + \cosh a + 2\cos\frac{\pi}{n}.$$

## References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Thank you for your attention!